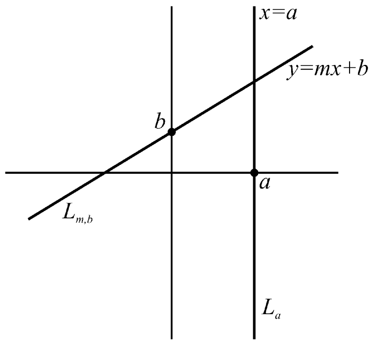
1 Definition and Models of Incidence Geometry

Definition (geometry)
A geometry is a set $\mathcal{S}$ of points and a set $\mathcal{L}$ of lines together with relationships between the points and lines.

1. Find four points which belong to set $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \,|\, x = \sqrt{5}\}$.
[for examle $P(\sqrt{5},0), Q(\sqrt{5},7), R(\sqrt{5},\sqrt{5}), M(\sqrt{5},\frac{-3}{2})$]

2. Let $\mathcal{M} = \{(x,y) \,|\, x,y \in \mathbb{R}, x < 0\}$.
(i) Find three points which belong to set $\mathcal{N} = \{(x,y) \in \mathcal{M} \,|\, x = \sqrt{2}\}$.
(ii) Are points $P(3,3)$, $Q(6,4)$ and $R(-2,\frac{4}{3})$ belong to set $\mathcal{L} = \{(x,y) \in \mathcal{M} \,|\, y = \frac{1}{3}x + 2\}$?
[$\mathcal{N} = \emptyset; P,Q \notin \mathcal{L}, R \in \mathcal{L}$]

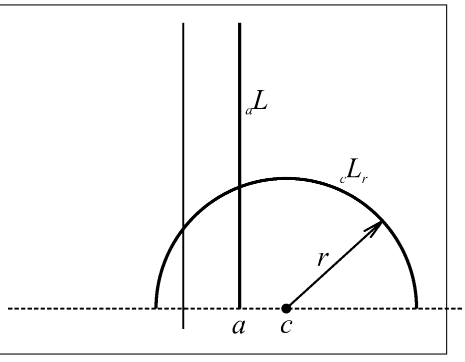


Definition (Cartesian Plane)
Let $\mathcal{L}_E$ be the set of all vertical and non-vertical lines, L_a and $L_{k,n}$ where
vertical lines:
$L_a = \{(x,y) \in \mathbb{R}^2 \,|\, x = a,\ a$ is fixed real number$\}$,
non-vertical lines:
$L_{k,n} = \{(x,y) \in \mathbb{R}^2 \,|\, y = kx + n,\ k$ and n are fixed real numbers$\}$.
The model $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ is called the Cartesian Plane.

3. Let $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ denote Cartesian Plane.
(i) Find three different points which belong to Cartesian vertical line L_7.
(ii) Find three different points which belong to Cartesian non-vertical line $L_{15,\sqrt{2}}$.
[for example $A(7,\frac{3}{2}), B(7,-4), C(7,0)$; $P(1,15+\sqrt{2}), Q(0,\sqrt{2}), R(-\sqrt{2}/15,0)$]

4. Let P be some point in Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$. Show that point P cannot lie simultaneously on both L_a and $L_{a'}$ (where $a \neq a'$). [suppose the contrary, $P \in L_a, P \in L_{a'} \Rightarrow a = a'$, contradiction]

Definition (Poincaré Plane.)
Let $\mathcal{L}_H$ be the set of all type I and type II lines, ${}_aL$ and ${}_pL_r$ where
type I lines:
${}_aL = \{(x,y) \in \mathbb{H} \,|\, x = a,\ a$ is a fixed real number$\}$,
type II lines:
${}_pL_r = \{(x,y) \in \mathbb{H} \,|\, (x-p)^2 + y^2 = r^2,\ p$ and n are fixed real numbers$\}$,
$\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\}$.
The model $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ will be called the Poincaré Plane.



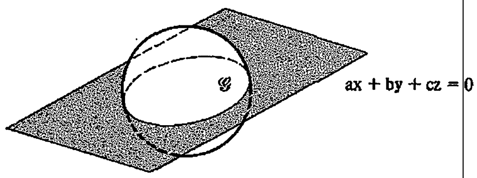
5. Find the Poincaré line through
(i) points $P(1,2)$ and $Q(3,4)$; [${}_5L_{2\sqrt{5}}$; ${}_5L_{\sqrt{10}}$]
(ii) points $M(1,2)$ and $N(3,4)$. [GeoGebra: $(x-5)\hat{}2 + y\hat{}2 = (2 * sqrt(5))\hat{}2$, $x = 1$, $x = 3$...]

6. Let P and Q denote two different points in Cartesian Plane $\mathbb{C} = \{R^2, \mathcal{L}_E\}$. Show that it is not possible for P and Q to lie simultaneously on two distinct lines L_a and $L_{k,n}$. [suppose the contrary]

Definition (unit sphere, plane)
The unit sphere in $\mathbb{R}^3$ is $S^2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$. A plane in $\mathbb{R}^3$ is a set of the form $\{(x,y,z) \in \mathbb{R}^3 \,|\, ax + by + cz = d$, where a,b,c,d are fixed real numbers, and not all of a,b,c are zero$\}$.
[GeoGebra: $x\hat{}2 + y\hat{}2 + z\hat{}2 = 1$, $x + y - (1/2) * z = 0$, find intersection]

Definition (great circle)
A great circle, $\mathcal{G}$, of the sphere S^2 is the intersection of S^2 with a plane through the origin. Thus $\mathcal{G}$ is a great circle if there are $a,b,c \in \mathbb{R}$ not all zero, with

$$\mathcal{G} = \{(x,y,z) \in S^2 \,|\, ax+by+cz=0\}.$$



7. Find a spherical line (great circle) through

(i) points $P(\frac{1}{2},\frac{1}{2},\sqrt{\frac{1}{2}})$ and $Q(1,0,0)$; [$\mathcal{G} = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2=1, -\sqrt{2}y+z=0\}$]

(ii) points $M(0,\frac{1}{2},\frac{\sqrt{3}}{2})$ and $N(0,-1,0)$. [$\mathcal{G} = \{(0,y,z) \in \mathbb{R}^3 \,|\, y^2+z^2=1\}$]

[GeoGebra: $x\text{^}2+y\text{^}2+z\text{^}2=1$, (1/2,1/2,sqrt(1/2)), (1,0,0), -sqrt(2)*y+z=0]

Definition (collinear set of points)
A set of points $\mathcal{P}$ is collinear if there is a line ℓ such that $\ell \subseteq \mathcal{P}$. $\mathcal{P}$ is non-collinear if $\mathcal{P}$ is not a collinear set.

8. Show by example that there are (at least) three non-collinear points in the Cartesian Plane.
[for example $P(7,3), Q(7,-2) \in L_7$, $R(-1,0) \notin L_7$, $P,Q \notin$ non-vertical line]

9. Verify that $P(x_1,y_1)$ and $Q(x_2,y_2)$ do lie on ${}_pL_r$, where p and r are given by

$$p = \frac{y_2^2-y_1^2+x_2^2-x_1^2}{2(x_2-x_1)}, \qquad r = \sqrt{(x_1-p)^2+y_1^2}.$$

[$(x_1-p)^2+y_1=r^2$, $(x_2-p)^2+y_2=r^2$]

10. Let P and Q denote two different points in Cartesian Plane $\mathbb{C} = \{R^2, \mathcal{L}_E\}$ which do not belong to the same vertical line. Show that P and Q cannot lie simultaneously on both $L_{k,n}$ and $L_{m,b}$.
[suppose the contrary, $P,Q \in L_{k,n} \Rightarrow k = \frac{y_2-y_1}{x_2-x_1}, n = y_1 - kx_1$...]

11. Prove that if P and Q are distinct points in $\mathbb{H}$ then they cannot lie simultaneously on both ${}_aL$ and ${}_pL_r$. [suppose the contrary, $P,Q \in {}_aL \Rightarrow x_1=x_2=a$; $P,Q \in {}_pL_r \Rightarrow y_1=y_2$...]

12. Show by example that there are (at least) three non-collinear points in the Poincaré Plane.
[for example $P(7,1), Q(7,-2) \in {}_7L$, $R(8,1) \notin {}_7L$, $P,Q \notin {}_pL_r$]

Definition (Riemann Sphere)
Let $\mathcal{L}_R$ be the set of great circles on S^2. The model $\mathcal{R} = \{S^2, \mathcal{L}_R\}$ is called the Riemann Sphere.

13. Explain, is it possible and are there two points on Riemann Sphere which lie simultaneously on two different spherical lines (great circles). If such two points exist, write them down, and find what are spherical lines which goes through that two points.
[$N(0,0,1)$, $S(0,0,1)$, $\{(x,y,z) \in S^2 \,|\, y=0\}$, $\{(x,y,z) \in S^2 \,|\, x=0\}$]

Definition (abstract geometry)
An abstract geometry $\mathcal{A}$ consists of a set $\mathcal{S}$, whose elements are called points, together with a collection $\mathcal{L}$ of non-empty subsets of $\mathcal{S}$, called lines, such that:
(i) For every two points $A,B \in \mathcal{S}$ there is a line $\ell \in \mathcal{L}$ with $A \in \ell$ and $B \in \ell$.
(ii) Every line has at least two points.

14. Show that the Cartesian Plane $\mathcal{C} = \{\mathbb{R}^2, \mathcal{L}_E\}$ is an abstract geometry. [$1^\circ\, x_1 = x_2$; $2^\circ\, x_1 \neq x_2$]

15. Show that the Poincaré Plane $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H\}$ is an abstract geometry. [$1^\circ\, x_1 = x_2$; $2^\circ\, x_1 \neq x_2$]

16. Show that the Riemann Sphere $\mathcal{R} = \{S^2, \mathcal{L}_R\}$ is an abstract geometry. [$ax_1+by_1+cz_1=0$...]